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Note

A Counterexample to a Conjecture of Abbott

PAUL LEMKE

Institute for Mathematics and Its Applications, University of Minnesota, Minneapolis, Minnesota 55455

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Arguments using elementary number theory are used to construct counterexamples to a conjecture of Abbott. © 1989 Academic Press, Inc.

In a 1980 note [1] H. L. Abbott proved a conjecture of Erdös and Silverman [2] stating that for some positive constant c, every set of n points in the Euclidean plane contains a subset of more than $cn^{1/2}$ points which does not contain the vertices of a right triangle. In particular, Abbott showed that this conjecture is true for all $c<1$ while Erdös and Silverman had already shown that it is false when $c \geqslant 2$. Abbott later [3] made the following conjecture: "For each positive integer n define $S_n \subset \mathbf{R}^2$ by

$$S_n = \{(x, y) \mid x, y \in \mathbf{Z}, \quad 0 \leqslant x \leqslant 2n, 0 \leqslant y \leqslant 2n, x \equiv y \pmod 2\},$$

where $\mathbf{Z}$ is the set of integers. Then any subset of S_n of more than $2n+1$ points contains the vertices of a right triangle." If true, this would imply that $c < 2^{1/2}$ in the Erdös–Silverman conjecture.

In this note I will show that Abbott's conjecture is false for all but a finite number of n, provided that one makes a reasonable (although unproven) assumption about the distribution of twin primes (primes that differ by only two). Moreover, specific counterexamples can easily be constructed by consulting a table of primes.

In particular, I will show that if we choose as a subset of S_n the set A_n defined by

$$A_n = I_n \cup J_n \cup K_n,$$

where

$$I_n = \{(2i, 0) \mid 2 \leqslant i \leqslant n, i \in \mathbf{Z}\},$$

$$J_n = \{(0, 2j) \mid 1 \leqslant j \leqslant n, j \in \mathbf{Z}\},$$

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and

$$K_n = \{(k, k+2) \mid n \leqslant k \leqslant 2n-3, k \in Q\},$$

where Q is the set of odd integers q such that q and $q+2$ each have exactly one distinct prime divisor, then the set A_n does not contain the vertices of a right triangle, but $|A_n| > 2n+1$ for some n.

Proof. If two of the vertices of a right triangle lie in I_n, then the third vertex lies either on one of the two lines passing through one of the two vertices and perpendicular to the line segment joining them, or else on the circle whose diameter is that segment. The circle is too small (radius $\leqslant n-1$) to intersect J_n or K_n, and clearly intersects I_n only at the original two vertices. The points lying on the two lines must have positive even x coordinate values, so that the lines miss J_n and K_n, and again intersect I_n at only the two original vertices. Thus the third vertex cannot be in A_n. Very similar arguments can be made over the case where the pair of vertices lie in either J_n or K_n, where in the latter case we note that the two lines have equations of the form $x + y = a$ where $a \geqslant 2n+2$, so that there is no intersection of the lines with I_n or J_n. We conclude that if A_n contains the vertices of a right traingle, there must be one vertex in each of the sets I_n, J_n, and K_n. Let these vertices be $(2i, 0)$, $(0, 2j)$, and $(k, k+2)$, respectively. We then consider the three cases:

Case I. The right angle is at $(k, k+2)$. For this case, $((k, k+2) - (2i, 0))((k, k+2) - (0, 2j))^{\mathrm{T}}$ must equal 0. Omitting the algebraic calculations, we obtain

$$i = k - j + 2 - \frac{2(j-1)}{k},$$

which implies $k \mid j-1$, since k is odd. But since $k \geqslant n > j-1 \geqslant 0$, this implies $j=1$ so that $i = k+1$. But $i \leqslant n$ and $k+1 > n$, so this is impossible.

Case II. The right angle is at $(0, 2j)$. Here, we derive

$$i = j - \frac{2j(j-1)}{k},$$

implying $k \mid j(j-1)$, since k is odd. But since k has only one distinct prime divisor, which cannot divide both j and $j-1$, k must be relatively prime to one of them and, therefore, must divide the other. If $k \mid j-1$ then we must have $j=1$ as before, so that $i=1$. But since $(2i, 0) \notin I_n$ for $i=1$, this is impossible. If $k \mid j$ then, since $k \geqslant j \geqslant 1$, we must have $k = n = j$, so that $i = 2-n$, which is impossible for $n \geqslant 1$.

Case III. The right angle is at $(2i, 0)$. We get

$$j = i - \frac{2i(i+1)}{k+2},$$

implying $k+2 \mid i(i+1)$, since $k+2$ is odd. Since $k+2$ is divisible by only one distinct prime, we again can conclude that $k+2$ divides either i or $i+1$. But since $k+2 \geqslant n+2 > i+1 > i > 0$, this is impossible.

Since $|A_n| = 2n - 1 + |K_n|$, we have $|A_n| > 2n+1$ whenever $|K_n| \geqslant 3$, and since $|K_n| = |Q \cap \{n, ..., 2n-3\}|$ and $Q = \{3, 5, 7, 9, 11, 17, 23, 25, 27, 29, ...\}$, we immediately see that A_n provides a counterexample to Abbott's conjecture for $n = 7$. ∎

Remarks. Straightforward calculations like those used for the three cases above show that the set $B_n = A_n \cup \{(3, 1)\} - \{(5, 7)\}$ is also right triangle free, and for $n = 6$ this gives the smallest known counterexample to Abbott's conjecture, illustrated in Fig. 1. Also, I can prove that this subset is maximal with the no-right-triangle property whenever $K_n \neq \phi$.

A reasonable assumption about the distribution of twin primes is that their density is proportional to $1/\ln^2 n$ for large n, and since this density dominates that of higher prime powers as $n \to \infty$, we expect that $|K_n| \sim cn/\ln^2 n$ for some constant c. Thus it appears that $|B_n| > 2n+1$ for all but a finite number of n, in fact for all $n \geqslant 6$. However, although (apparently) $\lim_{n \to \infty}(|B_n| - (2n+1)) = \infty$, we also have $\lim_{n \to \infty} |B_n|/(2n+1) = 1$, so that the ratio of the size of the no-right-triangle set constructed by this

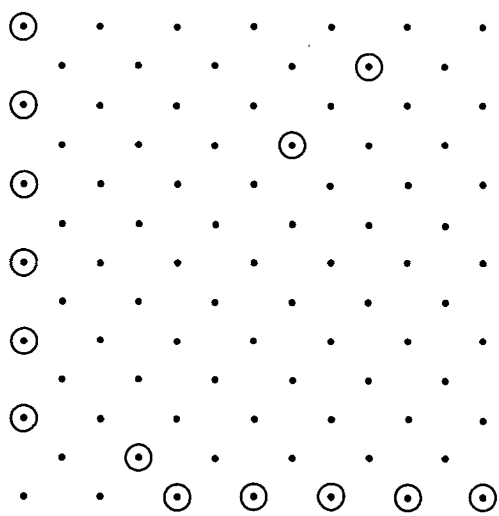

FIG. 1. The smallest known counterexample to Abbott's conjecture.

scheme to the conjectured upper bound reaches a maximum (at $n=7$, where it is $\frac{17}{15} \cong 1.133$) and then decreases back towards 1. Moreover, I do not see any similar scheme that will keep the ratio bounded away from 1 as $n \to \infty$. Perhaps Abbott's example is asymptotically valid as $n \to \infty$.

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References

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